Web-based Supplementary Materials for Ordered Multiple Comparisons with the Best and their Applications to Dose-Response Studies

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1. Web Appendix A

Proof. (Theorem 1) To prove Theorem 1 it has to be shown that for all $c_j \in \mathbb{R},\ j \in I_{k-1}$, and $i \in I_{k-1}$ it holds

$$\sup_{(\boldsymbol{\mu},\sigma)\in\Theta_i} P_{\boldsymbol{\mu},\sigma}(\bigcap_{j=1}^{i}\{T_j(\boldsymbol{X},S) \geq c_j\}) = P_{\mathbf{0},1}(\max_{r=i+1,\ldots,k}(\lambda_r+\lambda_i)^{-1/2}(X_r - X_i)/S \geq c_i).$$

Let $i \in I_{k-1}$ be fixed. Recalling that $T_i(\boldsymbol{X},S) = \max_{r=i+1,\ldots,k}(\lambda_r+\lambda_i)^{-1/2}(X_r - X_i - \epsilon)/S$, we get

$$\begin{aligned} & P_{\boldsymbol{\mu},\sigma}(\bigcap_{j=1}^{i}\{T_j(\boldsymbol{X},S) \geq c_j\}) \\ = \;& P_{\mathbf{0},1}(\bigcap_{j=1}^{i}\bigcup_{r=j+1}^{k}\{X_r - X_j + (\mu_r - \mu_j - \epsilon)/\sigma \geq c_j S\sqrt{\lambda_r+\lambda_j}\}). \end{aligned} \tag{1}$$

Let $(\boldsymbol{\mu},\sigma) \in \Theta_i = \{(\boldsymbol{\mu},\sigma) \in \Theta : \mu_{(k)} - \mu_i \leq \epsilon \text{ and } \mu_{(k)} - \max_{j=1,\ldots,i-1}\mu_j > \epsilon\}$. W.l.o.g. we can assume that $\mu_{(k)} = \max_{j=1,\ldots,k}\mu_j = 0$. For each $(\boldsymbol{\mu},\sigma) \in \Theta_i$ it then holds $\mu_j < -\epsilon$, for all $j = 1,\ldots,i-1$ and $\mu_i \in [-\epsilon, 0]$. The probability expression in (1) is obviously non-increasing in μ_1 and approaches $P_{\mathbf{0},1}(\bigcap_{j=2}^{i}\bigcup_{r=j+1}^{k}\{X_r - X_j + (\mu_r - \mu_j - \epsilon)/\sigma \geq c_j S\sqrt{\lambda_r+\lambda_j}\})$ if μ_1 tends to $-\infty$. Repeating this argument for all μ_j with $j < i$, we get the upper bound

$$P_{\mathbf{0},1}(\bigcup_{r=i+1}^{k}\{X_r - X_i + (\mu_r - \mu_i - \epsilon)/\sigma \geq c_i S\sqrt{\lambda_r+\lambda_i}\}).$$

This bound is non-increasing in μ_i and non-decreasing in μ_r for all $r = i+1, \ldots, k$, thus we get the final upper bound

$$P_{\mathbf{0},1}(\bigcup_{r=i+1}^{k} \{X_r - X_i \geq c_i S\sqrt{\lambda_r + \lambda_j}\}), \tag{2}$$

which is identical to right hand side of the equation given in Theorem 1. Let $a < -\epsilon$ be fixed and let $\boldsymbol{\mu}^* = (a, \ldots, a, -\epsilon, 0, \ldots, 0)$ with $\mu_i = -\epsilon$. Noting that $\lim_{\sigma \to 0} P_{\boldsymbol{\mu}^*,\sigma}(\bigcap_{j=1}^{i}\{T_j(\boldsymbol{X}, S) \geq c_j\})$ is equal to the lower bound (2) and that $\{(\boldsymbol{\mu}^*, \sigma) : \sigma \in (0, \infty)\} \subseteq \Theta_i$, the proof of Theorem 1 is complete.

2. Web Appendix B

Proof. (Lemma 1) In order to prove Lemma 1 we have to show that

$$\bigcap_{j=1}^{i}\{T_i(\boldsymbol{X}, S) \geq c_j\} = \bigcap_{j=1}^{i}\{T_i'(\boldsymbol{X}, S) \geq c_j\},$$

that is

$$\{\min_{j=1,\ldots,i} \max_{r \in I_k : r \neq j}(X_r - X_j - \epsilon - \sqrt{\lambda_r + \lambda_j} S) \geq 0\} = \{\min_{j=1,\ldots,i} \max_{r=j+1,\ldots,k}(X_r - X_j - \epsilon - \sqrt{\lambda_r + \lambda_j} S) \geq 0\}.$$

Let $\gamma_{r,j}(s) = \epsilon + \sqrt{\lambda_r + \lambda_j} s c_j$, $r \in I_k$, $j \in I_{k-1}$. Under the assumptions of Lemma 1 we get

$$\gamma_{r,p}(s) + \gamma_{p,q}(s) \geq \gamma_{r,q}(s) \text{ for all } s > 0,\ r \in I_k,\ p, q \in I_{k-1} \text{ with } p < \min(r, q). \tag{3}$$

Thus it is sufficient to prove that for each $\boldsymbol{x} \in \mathbb{R}^k$ and $s > 0$

$$\min_{j=1,\ldots,i} \max_{r \in I_k : r \neq j}(x_r - x_j - \gamma_{r,j}(s)) \geq 0 \tag{4}$$

holds if and only if

$$\min_{j=1,\ldots,i} \max_{r=j+1,\ldots,k}(x_r - x_j - \gamma_{r,j}(s)) \geq 0. \tag{5}$$

Obviously (5) implies (4), thus it remains to prove that (4) implies (5). Let $m(j) = \max\{r \in I_{k-1}, r \neq j : x_r - x_j \geq \gamma_{r,j}(s)\}$, then (4) implies (5) if $m(j) > j$ for all $j = 1, \ldots, i$.

Clearly we have $m(1) > 1$. Now assume we have shown $m(j) > j$ for all $j = 1, \dots, j'-1$, for some $j' \leq i$.

Suppose $m(j') < j'$. Then we have $m(m(j')) > m(j')$, thus (4) implies $x_{m(m(j'))} - x_{m(j')} \geq \gamma_{m(m(j')),m(j')}(s)$. Additionally we have $x_{m(j')} - x_{j'} \geq \gamma_{m(j'),j'}(s)$, thus $x_{m(m(j'))} - x_{j'} \geq \gamma_{m(m(j')),m(j')}(s) + \gamma_{m(j'),j'}(s)$. Inequality (3) yields $x_{m(m(j'))} - x_{j'} \geq \gamma_{m(m(j')),j'}(s)$. But this is a contradiction to the maximality of $m(j')$, thus $m(j')$ can not be less than j' so that the definition of $m(j')$ yields $m(j') > j'$. By induction we finally get $m(j) > j$, for all $j = 1, \dots, i$.

3. Web Appendix C

Proof. (Theorem 3) To prove Theorem 3 we have to show that

(a) $\pi^*(\alpha, \epsilon, \delta, \boldsymbol{\lambda}) \geq \inf_{(\boldsymbol{\mu},\sigma)\in\Theta} P_{\boldsymbol{\mu},\sigma}(\{i \in I_k : T_i(\boldsymbol{X}, S) \geq c_i\} \supseteq B(\boldsymbol{\mu}, \sigma)) = \min_{\ell \in I_k} \pi_\ell(\boldsymbol{\lambda})$,

(b) $\pi^*(\alpha, \epsilon, \delta, \boldsymbol{\lambda}) = \pi_k(\boldsymbol{\lambda})$ for all $\boldsymbol{\lambda}$ with $\lambda_1 = \cdots = \lambda_k$,

where

$$\pi^*(\alpha, \epsilon, \delta, \boldsymbol{\lambda}) = \inf_{(\boldsymbol{\mu},\sigma)\in\Theta} \pi(\boldsymbol{\mu}, \sigma | \alpha, \epsilon, \delta, \boldsymbol{\lambda}),$$

$$\pi(\boldsymbol{\mu}, \sigma | \alpha, \epsilon, \delta, \boldsymbol{\lambda}) = P_{\boldsymbol{\mu},\sigma}\left(L_{1-\alpha}(\boldsymbol{X}, S) \notin B(\boldsymbol{\mu}, \sigma)\right),$$

and

$$\pi_\ell(\boldsymbol{\lambda}) = P_{\mathbf{0},1}((\lambda_\ell + \lambda_i)^{-1/2}(X_\ell - X_i + \delta)/S \geq c_i \text{ for all } i \in I_k \setminus \{\ell\}).$$

W.l.o.g. assume $\max_{j=1,\dots,k} \mu_j = 0$. For each $\ell \in I_k \setminus B(\boldsymbol{\mu}, \sigma)$ we have

$$\begin{aligned} \{L_{1-\alpha}(\boldsymbol{X}, S) \notin B(\boldsymbol{\mu}, \sigma)\} &\supseteq \bigcap_{i \in B(\boldsymbol{\mu},\sigma)} \{T_i(\boldsymbol{X}, S) \geq c_i\} \\ &\supseteq \bigcap_{i \in B(\boldsymbol{\mu},\sigma)} \{\frac{X_\ell - X_i - \epsilon}{S\sqrt{\lambda_\ell + \lambda_i}} \geq c_i\}. \end{aligned}$$

As a consequence we get with $M_\ell = \{(\boldsymbol{\mu}, \sigma) : \mu_\ell = 0\}$ that

$$\pi^*(\alpha, \epsilon, \delta, \boldsymbol{\lambda}) \geq \min_{\ell \in I_k} \inf_{(\boldsymbol{\mu},\sigma)\in M_\ell} P_{\boldsymbol{\mu},\sigma}\Big(\bigcap_{i \in B(\boldsymbol{\mu},\sigma)} \Big\{\frac{X_\ell - X_i - \epsilon}{S\sqrt{\lambda_\ell + \lambda_i}} \geq c_i\Big\}\Big). \tag{6}$$

Straightforward monotonicity arguments yield that the right hand side of (6) is given by $\min_{\ell \in I_k} \pi_\ell(\boldsymbol{\lambda})$, thus we get part (a) of Theorem 3. To prove part (b) let $\lambda_1 = \cdots = \lambda_k$ and let $\alpha < 1/2$. Furthermore let $\boldsymbol{\mu}^* = \boldsymbol{\mu}^*(\sigma)$ be the vector with $\mu_k^* = 0$ and $\mu_j^* = -\epsilon - \delta\sigma$ for all $j \in I_{k-1}$. Taking into account that $B(\boldsymbol{\mu}^*, \sigma) = I_{k-1}$ we obtain

$$\begin{aligned}
\pi(\boldsymbol{\mu}^*, \sigma, |\alpha, \epsilon, \delta, \boldsymbol{\lambda}) &= P_{\boldsymbol{\mu}^*,\sigma}(L_{1-\alpha}(\boldsymbol{X}, S) = k) \\
&= P_{\boldsymbol{\mu}^*,\sigma}\Big(\bigcap_{i \in I_{k-1}} \{\max_{r=i+1,\ldots,k} X_r \geq X_i + \epsilon + c_i S\sqrt{2\lambda_1}\}\Big) \\
&= P_{\boldsymbol{\mu}^*,\sigma}\Big(\bigcap_{i \in I_{k-1}} \{X_k \geq X_i + \epsilon + c_i S\sqrt{2\lambda_1}\}\Big) \\
&= \pi_k(\boldsymbol{\lambda}),
\end{aligned}$$

where the third equation holds, because all c_i's are positive if $\alpha < 1/2$. In the balanced case the λ_i's are all equal, the distribution of $\boldsymbol{X} - \boldsymbol{\mu}$ is permutation invariant and the critical values c_i are non-increasing in i (cf. Remark 1). Keeping this in mind, it is easy to prove that $\min_{\ell \in I_{k-1}} \pi_\ell(\boldsymbol{\lambda}) \geq \pi_k(\boldsymbol{\lambda})$, which finally yields part (b) of Theorem 3.