1. Web Appendix A

Proof. (Theorem 1) To prove Theorem 1 it has to be shown that for all \( c_j \in \mathbb{R}, j \in I_{k-1} \), and \( i \in I_{k-1} \) it holds

\[
\sup_{(\mu, \sigma) \in \Theta_i} P_{\mu, \sigma}(\bigcap_{j=1}^{i} \{ T_j(\mathbf{X}, S) \geq c_j \}) = P_{0,1}(\max_{r=i+1, \ldots, k} (\lambda_r + \lambda_i)^{-1/2}(X_r - X_i)/S \geq c_i).
\]

Let \( i \in I_{k-1} \) be fixed. Recalling that \( T_i(\mathbf{X}, S) = \max_{r=i+1, \ldots, k} (\lambda_r + \lambda_i)^{-1/2}(X_r - X_i - \epsilon)/S \), we get

\[
P_{\mu, \sigma}(\bigcap_{j=1}^{i} \{ T_j(\mathbf{X}, S) \geq c_j \})
= P_{0,1}(\bigcap_{j=1}^{i} \bigcup_{r=j+1}^{k} \{ X_r - X_j + (\mu_r - \mu_j - \epsilon)/\sigma \geq c_j S \sqrt{\lambda_r + \lambda_j} \}).
\] (1)

Let \( (\mu, \sigma) \in \Theta_i = \{ (\mu, \sigma) \in \Theta : \mu_{(k)} = \max_{j=1, \ldots, k} \mu_j \leq \epsilon \} \). W.l.o.g. we can assume that \( \mu_{(k)} = \max_{j=1, \ldots, i-1} \mu_j = 0 \). For each \( (\mu, \sigma) \in \Theta_i \) it then holds \( \mu_j < -\epsilon \), for all \( j = 1, \ldots, i-1 \) and \( \mu_i \in [-\epsilon, 0] \). The probability expression in (1) is obviously non-increasing in \( \mu_1 \) and approaches \( P_{0,1}(\bigcap_{j=2}^{i} \bigcup_{r=j+1}^{k} \{ X_r - X_j + (\mu_r - \mu_j - \epsilon)/\sigma \geq c_j S \sqrt{\lambda_r + \lambda_j} \}) \) if \( \mu_1 \) tends to \( -\infty \). Repeating this argument for all \( \mu_j \) with \( j < i \), we get the upper bound

\[
P_{0,1}(\bigcup_{r=i+1}^{k} \{ X_r - X_i + (\mu_r - \mu_i - \epsilon)/\sigma \geq c_i S \sqrt{\lambda_r + \lambda_i} \}).
\]
This bound is non-increasing in $\mu_i$ and non-decreasing in $\mu_r$ for all $r = i + 1, \ldots, k$, thus we get the final upper bound

$$P_{0,1}(\bigcup_{r=i+1}^{k} \{X_r - X_i \geq c_i S \sqrt{\lambda_r + \lambda_j}\}),$$

which is identical to right hand side of the equation given in Theorem 1. Let $a < -\epsilon$ be fixed and let $\mu^* = (a, \ldots, a, -\epsilon, 0, \ldots, 0)$ with $\mu_i = -\epsilon$. Noting that $\lim_{\sigma \to 0} P_{\mu^*, \sigma}(\bigcap_{j=1}^{i} \{T_j(X, S) \geq c_j\})$ is equal to the lower bound (2) and that $\{(\mu^*, \sigma) : \sigma \in (0, \infty)\} \subseteq \Theta_i$, the proof of Theorem 1 is complete.

\section{Web Appendix B}

\begin{proof}{(Lemma 1)}

In order to prove Lemma 1 we have to show that

$$\bigcap_{j=1}^{i} \{T_{i}(X, S) \geq c_j\} = \bigcap_{j=1}^{i} \{T'_{i}(X, S) \geq c_j\},$$

that is

$$\{\min_{j=1, \ldots, i} \max_{r \in I_k: r \neq j} (X_r - X_j - \epsilon - \sqrt{\lambda_r + \lambda_j} S) \geq 0\} = \{\min_{j=1, \ldots, i} \max_{r = j+1, \ldots, k} (X_r - X_j - \epsilon - \sqrt{\lambda_r + \lambda_j} S) \geq 0\}.$$

Let $\gamma_{r,j}(s) = \epsilon + \sqrt{\lambda_r + \lambda_j} s c_j$, $r \in I_k$, $j \in I_{k-1}$. Under the assumptions of Lemma 1 we get

$$\gamma_{r,p}(s) + \gamma_{p,q}(s) \geq \gamma_{r,q}(s) \text{ for all } s > 0, \ r \in I_k, \ p, q \in I_{k-1} \text{ with } p < \min(r, q).$$

Thus it is sufficient to prove that for each $x \in \mathbb{R}^k$ and $s > 0$

$$\min_{j=1, \ldots, i} \max_{r \in I_k: r \neq j} (x_r - x_j - \gamma_{r,j}(s)) \geq 0 \quad \quad \quad (4)$$

holds if and only if

$$\min_{j=1, \ldots, i} \max_{r = j+1, \ldots, k} (x_r - x_j - \gamma_{r,j}(s)) \geq 0. \quad \quad \quad (5)$$

Obviously (5) implies (4), thus it remains to prove that (4) implies (5). Let $m(j) = \max\{r \in I_{k-1}, r \neq j : x_r - x_j \geq \gamma_{r,j}(s)\}$, then (4) implies (5) if $m(j) > j$ for all $j = 1, \ldots, i$. 2
Clearly we have \( m(1) > 1 \). Now assume we have shown \( m(j) > j \) for all \( j = 1, \ldots, j' - 1 \), for some \( j' \leq i \).

Suppose \( m(j') < j' \). Then we have \( m(m(j')) > m(j') \), thus (4) implies \( x_{m(m(j'))} - x_{m(j')} \geq \gamma_{m(m(j'))},m(j')(s) \). Additionally we have \( x_{m(j')} - x_{j'} \geq \gamma_{m(j')},j'(s) \), thus \( x_{m(m(j'))} - x_{j'} \geq \gamma_{m(m(j'))},m(j')(s) + \gamma_{m(j')},j'(s) \). Inequality (3) yields \( x_{m(m(j'))} - x_{j'} \geq \gamma_{m(m(j'))},j'(s) \). But this is a contradiction to the maximality of \( m(j') \), thus \( m(j') \) can not be less than \( j' \) so that the definition of \( m(j') \) yields \( m(j') > j' \). By induction we finally get \( m(j) > j \), for all \( j = 1, \ldots, i \).

3. Web Appendix C

Proof. (Theorem 3) To prove Theorem 3 we have to show that

(a) \( \pi^*(\alpha, \epsilon, \delta, \lambda) \geq \inf_{(\mu, \sigma) \in \Theta} P_{\mu, \sigma}(\{i \in I_k : T_i(X, S) \geq c_i\} \supseteq B(\mu, \sigma)) = \min_{\ell \in I_k} \pi_\ell(\lambda) \),

(b) \( \pi^*(\alpha, \epsilon, \delta, \lambda) = \pi_k(\lambda) \) for all \( \lambda \) with \( \lambda_1 = \cdots = \lambda_k \),

where

\[
\pi^*(\alpha, \epsilon, \delta, \lambda) = \inf_{(\mu, \sigma) \in \Theta} \pi(\mu, \sigma | \alpha, \epsilon, \delta, \lambda),
\]

\[
\pi(\mu, \sigma | \alpha, \epsilon, \delta, \lambda) = P_{\mu, \sigma}(L_{1-\alpha}(X, S) \notin B(\mu, \sigma)),
\]

and

\[
\pi_\ell(\lambda) = P_{0,1}((\lambda_\ell + \lambda_i)^{-1/2}(X_\ell - X_i + \delta)/S \geq c_i \text{ for all } i \in I_k \setminus \{\ell\}).
\]

W.l.o.g. assume \( \max_{j=1,\ldots,k} \mu_j = 0 \). For each \( \ell \in I_k \setminus B(\mu, \sigma) \) we have

\[
\{L_{1-\alpha}(X, S) \notin B(\mu, \sigma)\} \supseteq \bigcap_{i \in B(\mu, \sigma)} \{T_i(X, S) \geq c_i\}
\]

\[
\supseteq \bigcap_{i \in B(\mu, \sigma)} \left\{\frac{X_\ell - X_i - \epsilon}{S\sqrt{\lambda_\ell + \lambda_i}} \geq c_i\right\}.
\]
As a consequence we get with \( M_\ell = \{ (\mu, \sigma) : \mu_\ell = 0 \} \) that

\[
\pi^*(\alpha, \epsilon, \delta, \lambda) \geq \min_{\ell \in I_k} \inf_{(\mu, \sigma) \in M_\ell} \min_{i \in B(\mu, \sigma)} \left\{ \frac{X_\ell - X_i - \epsilon}{S\sqrt{\lambda_\ell + \lambda_i}} \geq c_i \right\}.
\]  

(6)

Straightforward monotonicity arguments yield that the right hand side of (6) is given by \( \min_{\ell \in I_k} \pi(\lambda) \), thus we get part (a) of Theorem 3. To prove part (b) let \( \lambda_1 = \cdots = \lambda_k \) and let \( \alpha < 1/2 \). Furthermore let \( \mu^* = \mu^*(\sigma) \) be the vector with \( \mu^*_k = 0 \) and \( \mu^*_j = -\epsilon - \delta \sigma \) for all \( j \in I_{k-1} \). Taking into account that \( B(\mu^*, \sigma) = I_{k-1} \) we obtain

\[
\pi(\mu^*, \sigma, |\alpha, \epsilon, \delta, \lambda) = P_{\mu^*, \sigma}(L_{1-\alpha}(X, S) = k) = P_{\mu^*, \sigma}(\bigcap_{i \in I_{k-1}} \{ \max_{r=i+1, \ldots, k} X_r \geq X_i + \epsilon + c_i S\sqrt{2\lambda_1} \}) = P_{\mu^*, \sigma}(\bigcap_{i \in I_{k-1}} \{ X_k \geq X_i + \epsilon + c_i S\sqrt{2\lambda_1} \}) = \pi_k(\lambda),
\]

where the third equation holds, because all \( c_i \)'s are positive if \( \alpha < 1/2 \). In the balanced case the \( \lambda_i \)'s are all equal, the distribution of \( X - \mu \) is permutation invariant and the critical values \( c_i \) are non-increasing in \( i \) (cf. Remark 1). Keeping this in mind, it is easy to prove that \( \min_{\ell \in I_{k-1}} \pi(\lambda) \geq \pi_k(\lambda) \), which finally yields part (b) of Theorem 3.