Web-based Supplementary Materials for “A KPSS Test for Stationarity for Spatial Point Processes” by Yongtao Guan

Web Appendix A: The Intensity and Cumulant Functions
Let $D_n$ denote $[0, n_1] \times [0, n_2]$. For a Borel set $B \subset \mathbb{R}^2$, let $|B|$ denote the area of $B$, and $N(B)$ denote the number of events from $N$ that fall in $B$. We define the $k$th-order intensity and cumulant functions of $N$ as:

$$
\lambda_k(s_1, \cdots, s_k) = \lim_{|ds_i| \to 0} \left\{ \frac{E[N(ds_1) \cdots N(ds_k)]}{|ds_1| \cdots |ds_k|} \right\}, \quad i = 1, \cdots, k,
$$

$$
Q_k(s_1, \cdots, s_k) = \lim_{|ds_i| \to 0} \left\{ \frac{CUM[N(ds_1), \cdots, N(ds_k)]}{|ds_1| \cdots |ds_k|} \right\}, \quad i = 1, \cdots, k,
$$

respectively, where $s_i = (x_i, y_i)$, $ds$ is an infinitesimal region containing $s$ and $CUM(\cdot)$ is the cumulant measure (e.g. Brillinger, 1975). For the intensity functions, $\lambda_k(s_1, \cdots, s_k)|ds_1| \cdots |ds_k|$ is the approximate probability for $ds_1, \cdots, ds_k$ to each contain an event. For the cumulant functions, $Q_k(s_1, \cdots, s_k)$ describes the dependence among sites $s_1, \cdots, s_k$, where a close-to-zero value indicates near independence. Specifically, if $N$ is Poisson, then $Q_k(s_1, \cdots, s_k) = 0$ if at least two of $s_1, \cdots, s_k$ are different. The $k$-th order cumulant (intensity) function can be expressed as a linear function of the intensity (cumulant) functions up to the $k$-th order. See Daley and Vere-Jones (1988, p.147) for details.

Web Appendix B: Convergence of $T_n$
Define

$$
X_n(t_1, t_2) = (n_1n_2)^{-1/2} \left[ N(n_1t_1, n_2t_2) - n_1n_2t_1t_2\lambda \right].
$$

(B.1)
Observe that $S_n(u_1, u_2) = (n_1n_2)^{1/2}[X_n(u_1/n_1, u_2/n_2) - (u_1u_2)/(n_1n_2)X_n(1, 1)]$. The convergence of $T_n$ to $\int_0^1 \int_0^1 [V(t_1, t_2)]^2 dt_1 dt_2$ can be proved trivially by noting the following theorem.

**Theorem 1.** Let $N$ be a strongly mixing stationary spatial point process. Assume that the cumulant function exists up to the fourth order and satisfies the following conditions:

\[
\int \cdots \int |Q_k(s_1, \cdots, s_{k-1}, s_k)| ds_1 \cdots ds_{k-2} < C \quad \text{for } k = 2, 3, 4 \tag{B.2}
\]

\[
\int \cdots \int |Q_k(s_1, \cdots, s_{k-1}, s_k)| ds_1 \cdots ds_{k-1} < C \quad \text{for } k = 2, 3, 4 \tag{B.3}
\]

then for arbitrary $t_1, t_2 \in (0, 1]$, $\sigma^{-1}X_n(t_1, t_2)$ converges weakly to a Wiener process, where $X_n(t_1, t_2)$ is defined by (B.1).

**Proof.** See Corollary 7.2 of Ivanoff (1982).

The strong mixing condition assumed in the above is a standard assumption for deriving asymptotic distributions for both quantitative processes (e.g. Cressie 1993) and point processes (e.g. Daley and Vere-Jones 1988). In the spatial point process case, it is satisfied by a large class of models including the two examples considered in our simulation study and any Cox process model with a strongly mixing random intensity field. For a detailed discussion on mixing conditions, see Daley and Vere-Jones (1988). Conditions (B.2) and (B.3) can be verified directly for a given process. Specifically, they hold for the two examples used in our simulation.
Web Appendix C: Consistency of the Nonparametric Variance Estimator

By Taylor series expansion,
\[ \hat{\lambda}^2_n = \lambda^2 + 2\lambda(\hat{\lambda}_n - \lambda) + 2(\hat{\lambda}_n - \lambda)^2 \Rightarrow E(\hat{\lambda}^2_n) \approx \lambda^2 + 2(n_1n_2)^{-1}\sigma^2. \]

Thus
\[ E[\sigma^2_n] \approx \lambda^2 \int_{-n_1}^{n_1} \int_{-n_2}^{n_2} [g(l_1, l_2) - 1] I(l_1^2 + l_2^2 \leq m_n^2) dl_1 dl_2 + \lambda \rightarrow \sigma^2. \]

To establish the consistency, note that
\[ \sum_{s, s' \in D_n} I[(x - x')^2 + (y - y')^2 \leq m_n^2] - \hat{\lambda}^2_n \pi m_n^2 \]
\[ = \sum_{s, s' \in D_n} \left\{ I[(x - x')^2 + (y - y')^2 \leq m_n^2] - \pi m_n^2 \right\} \frac{(n_1 - |x - x'|)(n_2 - |y - y'|)}{n_1^2 n_2^2} - \hat{\lambda}_n \pi m_n^2 n_1 n_2. \]

The last term in the above converges to zero in probability. So we only need to show:
\[ VAR \left[ \sum_{s, s' \in D_n} \left\{ I[(x - x')^2 + (y - y')^2 \leq m_n^2] - \pi m_n^2 \right\} \frac{(n_1 - |x - x'|)(n_2 - |y - y'|)}{n_1^2 n_2^2} \right] \rightarrow 0. \]

To see this, note that the term in the variance has the general form
\[ T = \sum_{s, s' \in D_n} \phi(s, s'), \]
where \( s = (x, y), \phi \) is symmetric and \( D_n = [0, n_1] \times [0, n_2] \). From the expressions in Ripley (1988, p.30), we have
\[ VAR(T) = \int_{D_n} \int_{D_n} \phi(s_1, s_2) \phi(s_3, s_4) \left\{ d\alpha_4(s_1, s_2, s_3, s_4) - d\alpha_2(s_1, s_2) d\alpha_2(s_3, s_4) \right\} \]
\[ + \int_{D_n} \int_{D_n} \phi(s_1, s_2) \phi(s_1, s_3) d\alpha_3(s_1, s_2, s_3) \]
\[ + \int_{D_n} \phi(s_1, s_2)^2 d\alpha_2(s_1, s_2), \]
where $\alpha_k$ is the $k$th factorial moment measure (e.g., Stoyan and Stoyan 1994). All three terms converge to zero under condition (B.3) by observing $d\alpha_k(s_1, \ldots, s_k) = \lambda_k(x_1, \ldots, s_k)ds_1 \cdots ds_k$ and the fact that $\lambda_k$ can be written as a linear function of the cumulant functions up to order $k$. The derivations are tedious yet rather elementary. We thus omit their presentation.

**Web Appendix D: Consistency of the Test**

First, we study $\hat{\sigma}^2_n$. Note that

$$E(\hat{\sigma}^2_n) = \int \int \int \int \frac{\lambda_2[(x, y), (x', y')]}{(n_1 - |x - x'|)(n_2 - |y - y'|)} dx dx' dy dy'$$

$$- \frac{\pi m_n^2}{n_1 n_2^2} \int \int \int \lambda_2[(x, y), (x', y')] dx dx' dy dy'$$

$$- \frac{\pi m_n^2}{n_1 n_2^2} \int \int \lambda(x, y) dx dy + \frac{1}{n_1 n_2} \int \int \lambda(x, y) dx dy$$

$$\leq C m_n^2 n^{-2\gamma} + C m_n^2 (n_1 n_2)^{-1} n^\gamma + C n\gamma,$$

where the inequality in the above is due to conditions (10) and (11) in the text.

Let $\mu_n(u_1, u_2) = E[S_n(u_1, u_2)]$. Then,

$$\int_0^{n_1} \int_0^{n_2} [S_n(u_1, u_2)]^2 du_1 du_2$$

$$= \int_0^{n_1} \int_0^{n_2} [\mu_n(u_1, u_2)]^2 du_1 du_2$$

$$+ \frac{1}{n_1 n_2^2} \int_0^{n_1} \int_0^{n_2} [S_n(u_1, u_2) - \mu_n(u_1, u_2)]^2 du_1 du_2$$

$$+ \frac{2}{n_1 n_2} \int_0^{n_1} \int_0^{n_2} [S_n(u_1, u_2) - \mu_n(u_1, u_2)] \mu_n(u_1, u_2) du_1 du_2.$$
Define the three terms as $A_n$, $B_n$ and $C_n$, respectively. Note that
\[
E[S_n(u_1, u_2)] = \int_0^{u_2} \int_0^{u_1} \lambda(x, y) dx dy - \frac{u_1 u_2}{n_1 n_2} \int_0^{n_1} \int_0^{n_2} \lambda(x, y) dx dy
\]
\[
\to n^{\gamma} \left[ \int_0^{u_2} \int_0^{u_1} \lambda_0(x/n_1, y/n_2) dx dy - \frac{u_1 u_2}{n_1 n_2} \int_0^{n_1} \int_0^{n_2} \lambda_0(x/n_1, y/n_2) dx dy \right]
\]
\[
= (n_1 n_2)^{\gamma} \left[ \int_0^{u_2/n_1} \int_0^{u_1/n_2} \lambda_0(x, y) dx dy - \frac{u_1 u_2}{n_1 n_2} \int_0^{1} \int_0^{1} \lambda_0(x, y) dx dy \right].
\]
Thus
\[
A_n \to n^{2\gamma}(n_1 n_2) \int_0^{1} \int_0^{1} \left[ \int_0^{t_2} \int_0^{t_1} \lambda_0(x, y) dx dy - t_1 t_2 \int_0^{1} \int_0^{1} \lambda_0(x, y) dx dy \right] dtdt_2.
\]
Clearly $B_n > 0$. We would like to show that $(n_1 n_2)^{-1} n^{-2\gamma} B_n \to 0$ in probability. Thus $A_n$ is the dominating term of $A_n$, $B_n$ and $C_n$. To do so, we only need to show $(n_1 n_2)^{-1} n^{-2\gamma} E(B_n) \to 0$. Note that
\[
(n_1 n_2)^{-1} n^{-2\gamma} E(B_n)
\]
\[
= \frac{1}{n_1^3 n_2^3 n^{2\gamma}} \int_0^{n_1} \int_0^{n_2} \text{VAR}[S_n(u_1, u_2)] du_1 du_2
\]
\[
< \frac{2}{n_1^3 n_2^3 n^{2\gamma}} \int_0^{n_1} \int_0^{n_2} \text{VAR}[N(u_1, u_2)] du_1 du_2
\]
\[
+ \frac{2 \text{VAR}[N(n_1, n_2)]}{n_1^3 n_2^3 n^{2\gamma}} \int_0^{n_1} \int_0^{n_2} \frac{u_1^2 u_2^2}{n_1^2 n_2^2} du_1 du_2
\]
\[
= \frac{2}{n_1^3 n_2^3 n^{2\gamma}} \int_0^{n_1} \int_0^{n_2} \text{VAR}[N(u_1, u_2)] du_1 du_2 + \frac{2 \text{VAR}[N(n_1, n_2)]}{9n_1^2 n_2^2 n^{2\gamma}}.
\]
Furthermore, because of condition (12) in the text, we have
\[
\text{VAR}[N(u_1, u_2)]
\]
\[
= \int_0^{u_1} \int_0^{u_2} \{\lambda_2[(x, y), (x', y')]} - \lambda(x, y) \lambda(x', y') \} dx dy dx'dy' + \int_0^{u_1} \int_0^{u_2} \lambda(x, y) dx dy
\]
\[
\leq C n^{\gamma} \int_0^{u_1} \int_0^{u_2} \lambda(x, y) dx dy + \int_0^{u_1} \int_0^{u_2} \lambda(x, y) dx dy
\]
\[
\leq C(n^{\gamma} + 1) n^{\gamma} u_1 u_2.
\]
So that
\[(n_1 n_2)^{-1} n^{-2\gamma} E(B_n) \leq \frac{C n^{\gamma} (n^{\gamma} + 1)}{n_1 n_2 n^{2\gamma}} \to 0 \text{ if } \gamma > -2.\]

Thus
\[T_n > \frac{C n^{2\gamma} (n_1 n_2)}{m_n^2 n^{2\gamma} + C m_n^2 (n_1 n_2)^{-1} n^{\gamma} + C n^{\gamma}} \to \infty \text{ if } \gamma > -2.\]

Web Appendix E: Convergence of $T_n^*$

Let $X^i_n(t_1, t_2)$, $i = 1, \cdots, 4$, be the random processes defined in (B.1), where the four corners of the rectangular region, starting from the lower left and going clockwise toward the bottom right, are in turn defined as the four origins. The convergence of $T_n^*$ to the target limiting distribution follows trivially by noting the following relationships:

\[S^i_n(u_1, u_2) = (n_1 n_2)^{1/2} [X^i_n(u_1/n_1, u_2/n_2) - (u_1 u_2)/(n_1 n_2) X^i_n(1, 1)],\]
\[X^2_n(t_1, t_2) = X^1_n(t_1, 1) - X^1_n(t_1, 1 - t_2),\]
\[X^3_n(t_1, t_2) = X^4_n(1, 1) - X^4_n(1 - t_1, 1) - X^4_n(1, 1 - t_2) + X^4_n(1 - t_1, 1 - t_2),\]
\[X^4_n(t_1, t_2) = X^1_n(1, t_2) - X^1_n(1 - t_1, t_2).\]

Web Figure 1: The empirical pair correlation function (PCF) plot for the longleaf pine data.
References


