1 Derivation of ARE of the Mann-Whitney U test vs. the 2-sample t test under the alternative hypothesis in equation 6 (Section 2.2)

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We first consider the ARE of the Mann-Whitney U test vs. the 2-sample z test.

Suppose that $X_1, \ldots, X_m \sim N(\mu_0, \sigma^2)$ and $Y_1, \ldots, Y_n \sim N(\mu_0 + \mu, \sigma^2)$. We have that:

$$\theta = \Pr(X_i - Y_j < 0) = \Phi\{\mu/(\sigma\sqrt{2})\} \text{ or } \mu = \sqrt{2}\sigma\Phi^{-1}(\theta)$$

The test statistic for the 2-sample z test is $\bar{y} - \bar{x}$. It follows that

$$\begin{aligned} \partial E(\bar{y} - \bar{x})/\partial\theta]_{\theta=1/2} = \sqrt{2}\sigma\partial\Phi^{-1}(\theta)/\partial\theta]_{\theta=1/2} &= \sqrt{2}\sigma\sqrt{2\pi}\exp[\frac{1}{2}\{\Phi^{-1}(\theta)\}^2]_{\theta=1/2} \\ &= 2\sigma\sqrt{\pi} \equiv E_1 \end{aligned}$$

Furthermore,

$$\text{var}_0(\bar{y} - \bar{x}) = \sigma^2(1/m + 1/n) \equiv V_1$$

For the Mann-Whitney U test, the test statistic is $\hat{\theta}$ in (3). We have $E(\hat{\theta}) = \theta$. Hence, $\partial E(\hat{\theta})/\partial\theta = 1 \equiv E_2$. Furthermore, $\text{var}_0(\hat{\theta}) = (m + n + 1)/(12mn) \equiv V_2$. It follows that the ARE of the Mann-Whitney U Test vs. the two sample z-test (Kendall and Stuart, 1969) is

$$\begin{aligned} \text{ARE} &= (E_2^2/V_2)/(E_1^2/V_1) = \{(m + n + 1)/(12mn)\}^{-1}/[4\pi\sigma^2/\{\sigma^2(m + n)/(mn)\}] \\ &= (3/\pi)(m + n)/(m + n + 1) \rightarrow 3/\pi \quad \text{as} \quad m, n \rightarrow \infty \qquad \text{(A.1)} \end{aligned}$$

Since the 2-sample t test is asymptotically equivalent to the 2-sample z test (Kendall and Stuart, 1969), it follows that the ARE of the Mann-Whitney U test vs. the 2-sample t test is also $3/\pi$ in the case of normality. This confirms a previous result of Hodges and Lehmann (1956).

2 Derivation of $\text{var}(\hat{\theta}_A - \hat{\theta}_B)$ in equation 29, section 4.1

Using equation (16), we have:

$$\text{var}(\hat{\theta}_A) = \left[\theta_A(1 - \theta_A) + (m + n - 2)\left[\Phi_2\{\Phi^{-1}(\theta_A), \Phi^{-1}(\theta_A), 1/2\} - \theta_A^2\}\right]\right]/(mn) \qquad \text{(A.2)}$$

$$\text{var}(\hat{\theta}_B) = \left[\theta_B(1 - \theta_B) + (m + n - 2)\left[\Phi_2\{\Phi^{-1}(\theta_B), \Phi^{-1}(\theta_B), 1/2\} - \theta_B^2\}\right]\right]/(mn) \qquad \text{(A.3)}$$

To determine $\text{Cov}(\hat{\theta}_A, \hat{\theta}_B)$ we can write

$$\hat{\theta}_A = \sum_{i=1}^{m} U_{Ai}/(mn), \quad \hat{\theta}_B = \sum_{i=1}^{m} U_{Bi}/(mn) \quad \text{where} \quad U_{Ai} = \sum_{k=1}^{n} U_{Aik}, \quad U_{Bi} = \sum_{k=1}^{n} U_{Bik}$$

and $U_{Aik} = U(X_{A_i} - Y_{A_k}), U_{Bik} = U(X_{B_i} - Y_{B_k})$. We have

$$\begin{aligned} \text{Cov}(\hat{\theta}_A, \hat{\theta}_B) &= \frac{1}{m^2n^2} \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \text{Cov}(U_{Ai_1}, U_{Bi_2}) \qquad (A.4) \\ &= \frac{1}{m^2n^2} \left\{ \sum_{i=1}^{m} \text{Cov}(U_{Ai}, U_{Bi}) + \sum_{i_1 \neq i_2 = 1}^{m} \text{Cov}(U_{Ai_1}, U_{Bi_2}) \right\} \end{aligned}$$

Furthermore,

$$\begin{aligned} \text{Cov}(U_{Ai}, U_{Bi}) &= \sum_{k=1}^{n} \text{Cov}(U_{Aik}, U_{Bik}) + \sum_{k_1 \neq k_2 = 1}^{n} \text{Cov}(U_{Aik_1}, U_{Bik_2}) \qquad (A.5) \\ \text{Cov}(U_{Aik}, U_{Bik}) &= \Pr(X_{A_i} - Y_{A_k} < 0 \text{ and } X_{B_i} - Y_{B_k} < 0) - \theta_A\theta_B \\ &= \Pr(H_{XA_i} - H_{YA_k} < 0 \text{ and } H_{XB_i} - H_{YB_k} < 0) - \theta_A\theta_B \end{aligned}$$

where

$$\begin{aligned} H_{XA_i} &= \Phi^{-1}\{F_{XA}(X_{A_i})\}, \; H_{YA_k} = \Phi^{-1}\{F_{XA}(Y_{A_k})\} + \mu_A, \\ H_{XB_i} &= \Phi^{-1}\{F_{XB}(X_{B_i})\}, \; H_{YB_k} = \Phi^{-1}\{F_{XB}(Y_{B_k})\} + \mu_B \end{aligned}$$

and F_{XA}, F_{XB} are the c.d.f.'s of risk scores in the X group for risk prediction rules A and B, respectively.

Hence, $H_{YA_k} - H_{XA_i} \sim N(\mu_A, 2)$, $H_{YB_k} - H_{XB_i} \sim N(\mu_B, 2)$. Let $V_{Aik} = (H_{YA_k} - H_{XA_i} - \mu_A)/\sqrt{2}$, $V_{Bik} = (H_{YB_k} - H_{XB_i} - \mu_B)/\sqrt{2}$. It follows that $V_{Aik} \sim N(0,1)$ and $V_{Bik} \sim N(0,1)$. Thus,

$$\begin{aligned} &\Pr(H_{XA_i} - H_{YA_k} < 0 \text{ and } H_{XB_i} - H_{YB_k} < 0) \\ =\; &\Pr(V_{Aik} > -\mu_A/\sqrt{2} \text{ and } V_{Bik} > -\mu_B/\sqrt{2}) \\ =\; &\Pr(V_{Aik} < \mu_A/\sqrt{2} \text{ and } V_{Bik} < \mu_B/\sqrt{2}) \end{aligned}$$

Furthermore,

$$\begin{aligned} \text{Cov}(V_{Aik}, V_{Bik}) &= \frac{1}{2}\{\text{Cov}(H_{YA_k}, H_{YB_k}) + \text{Cov}(H_{XA_i}, H_{XB_i})\} \\ &= (\rho_y + \rho_x)/2 \end{aligned}$$

where $\rho_x = \text{corr}(H_{XA}, H_{XB}), \rho_y = \text{corr}(H_{YA}, H_{YB})$. Hence,

$$\begin{aligned} \text{Cov}(U_{Aik}, U_{Bik}) &= \Pr(V_{Aik} < \mu_A/\sqrt{2} \text{ and } V_{Bik} < \mu_B/\sqrt{2}) - \theta_A\theta_B \\ &= \Phi_2\{\mu_A/\sqrt{2}, \mu_B/\sqrt{2}, (\rho_x + \rho_y)/2\} - \theta_A\theta_B \qquad (A.6) \end{aligned}$$

We now consider $\text{Cov}(U_{Aik_1}, U_{Bik_2})$ in (A.5).

$$\text{Cov}(U_{Aik_1}, U_{Bik_2}) = \Pr(V_{Aik_1} < \mu_A/\sqrt{2} \text{ and } V_{Bik_2} < \mu_B/\sqrt{2}) - \theta_A\theta_B$$

However, $\text{Cov}(V_{Aik_1}, V_{Bik_2}) = \frac{1}{2}\ \text{Cov}(H_{XA_i}, H_{XB_i}) = \rho_x/2$. Thus,

$$\text{Cov}(U_{Aik_1}, U_{Bik_2}) = \Phi_2\left\{\mu_A/\sqrt{2}, \mu_B/\sqrt{2}, \rho_x/2\right\} - \theta_A\theta_B \tag{A.7}$$

It follows from (A.5)-(A.7) that

$$\begin{aligned} \text{Cov}(U_{Ai}, U_{Bi}) &= n\left[\Phi_2\left\{\mu_A/\sqrt{2}, \mu_B/\sqrt{2}, (\rho_x+\rho_y)/2\right\} - \theta_A\theta_B\right] \\ &+ n(n-1)\left[\Phi_2\left\{\mu_A/\sqrt{2}, \mu_B/\sqrt{2}, \rho_x/2\right\} - \theta_A\theta_B\right] \end{aligned} \tag{A.8}$$

We now need to determine $\text{Cov}(U_{Ai_1}, U_{Bi_2})$ in (A.4). We have

$$\text{Cov}(U_{Ai_1}, U_{Bi_2}) = \sum_{k=1}^{n}\ \text{Cov}(U_{Ai_1k}, U_{Bi_2k}) = n\text{Cov}(U_{Ai_1k}, U_{Bi_2k})$$

By symmetry, from (A.7), we have

$$\text{Cov}(U_{Ai_1}, U_{Bi_2}) = n\left[\Phi_2\left\{\mu_A/\sqrt{2}, \mu_B/\sqrt{2}, \rho_y/2\right\} - \theta_A\theta_B\right] \tag{A.9}$$

Therefore, if we substitute $\Phi^{-1}(\theta_A)$ for $\mu_A/\sqrt{2}$ and $\Phi^{-1}(\theta_B)$ for $\mu_B/\sqrt{2}$, then from (A.4), (A.8) and (A.9) we have:

$$\begin{aligned} \text{Cov}(\hat{\theta}_A, \hat{\theta}_B) &= \Big[\left[\Phi_2\left\{\Phi^{-1}(\theta_A), \Phi^{-1}(\theta_B), (\rho_x+\rho_y)/2\right\} - \theta_A\theta_B\right] \\ &+ (n-1)\left[\Phi_2\left\{\Phi^{-1}(\theta_A), \Phi^{-1}(\theta_B), \rho_x/2\right\} - \theta_A\theta_B\right] \\ &+ (m-1)\left[\Phi_2\left\{\Phi^{-1}(\theta_A), \Phi^{-1}(\theta_B), \rho_y/2\right\} - \theta_A\theta_B\right]\ \Big]/(mn) \end{aligned} \tag{A.10}$$

Upon combining (A.2), (A.3) and (A.10) we obtain the expression for $\text{var}(\hat{\theta}_A - \hat{\theta}_B)$ in equation 29.