Additional Material for

A Model-Based Approach for Making Ecological Inference From Distance Sampling Data

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Appendix A
Connections to Traditional Distance Sampling Estimators

If we assume that the underlying process that produces the locations, whether or not they are sighted, is a homogenous Poisson process with constant rate \( \lambda(s; \beta) = \lambda \), then the log-likelihood of \( S \) may be written

\[
\ell_{HP}(\eta, \lambda; S) = n \ln(\lambda) + \frac{K}{n} \sum_{i=1}^{n} \ln(g(s_i; \eta_k)) - \lambda \sum_{k=1}^{K} \int_{A} g(u; \eta_k) \, du. \tag{1}
\]

Historically, density is usually the main interest in line transect sampling. The expected total number of animals is \( \lambda|A| \), where \( |A| \) is the area of \( A \). Hence, we will use (1) to estimate the parameter \( \lambda \) and obtain an estimate of the total number of animals.

Perhaps the simplest detection function is the uniform. Assume, for all transects the detection function is given by

\[
g(s; w) = I(z(s) < w), \tag{2}
\]

where \( I(\cdot) \) is the indicator function and \( w \) is the known viewing width along the transect. Recall that \( z(s) \) is the perpendicular distance from \( s \) to the transect. The detection function (2) assumes all points are observed within the viewing width \( w \).

Substituting (2) into (1) and maximizing for \( \lambda \) we obtain the following maximum likelihood density estimator,

\[
\hat{\lambda} = \frac{n}{2wL}, \tag{3}
\]
where $L$ is the total length of all transects. Equation (3) is exactly the estimator found in the pioneering paper on line transect sampling by Hayne (1949). The estimator (3) is often the first one mentioned in monographs and texts on line transect sampling (e.g. Burnham, Anderson, and Laake 1980) because of its conceptual value. Note that historically (3) was derived under design-based principles by randomizing the locations of the line transects for a fixed set of points. It is interesting that we obtain the same estimator when considering the line fixed, but the points are generated by a homogeneous Poisson process. Next consider a general family of detectability functions

$$g(s; \alpha, \gamma) = \exp\{-[z(s)/\alpha]^{1/\gamma}\}, \quad (4)$$

where $\alpha > 0$ and $\gamma > 0$. Notice that as $\gamma \to 0$, then (4) becomes, essentially, (2), with $w = \alpha$. The family of detection functions in (4) covers many of the “key” functions listed in Buckland et al. (1993, p. 46 - 49), including the uniform, half-normal ($\gamma = 0.5$), exponential ($\gamma = 1$), and hazard detectability functions (the untruncated forms). Substituting (4) into (1) we obtain

$$\ell(\alpha, \gamma, \lambda; S) = n \ln(\lambda) - \ln(n!) - \sum_{i=1}^{n} [z_i/\alpha]^{1/\gamma} - \lambda 2L\alpha\gamma\Gamma(\gamma). \quad (5)$$

For fixed $\gamma$, the maximum likelihood estimator for $\alpha$ is

$$\hat{\alpha} = \left[ \sum_{i=1}^{n} z_i^{1/\gamma}/(n\gamma) \right]^{\gamma} \quad (6)$$

Substituting (6) into (5), then a maximum likelihood estimator of $\lambda$ for any fixed $\gamma$ is

$$\hat{\lambda} = \frac{n}{2L\gamma^{1-\gamma}(\sum_{i=1}^{n} z_i^{1/\gamma}/n)^\gamma \Gamma(\gamma)} \quad (7)$$

Special cases of (5) are the exponential detection function ($\gamma = 1$) with

$$\hat{\lambda} = \frac{n}{2L\bar{z}},$$

where $\bar{z} \equiv (1/n) \sum_{i=1}^{n} z_i$ is the average distance from the $n$ observations to the line transect, and the half-normal detectability function ($\gamma = 0.5$),

$$\hat{\lambda} = \frac{n}{L\sqrt{2\pi \bar{z}^2}}.$$
where \( \bar{z}^2 \equiv (1/n) \sum_{i=1}^n z_i^2 \) is the average distance-squared among each of the \( n \) observations and the line transect. This is the same estimator as derived by Buckland et al. (1993, p. 71). Substituting \( \hat{\alpha} \) in (6) and \( \hat{\lambda} \) in (7) back into (5) we obtain

\[
\ell_{GA}(\gamma; S) = (\gamma - 1) \ln(\gamma) - \gamma \ln \left( \sum_{i=1}^n z_i^{1/\gamma} / n \right) - \ln[\Gamma(\gamma)] - \gamma. \tag{8}
\]

The loglikelihood (8) can be minimized numerically to obtain the shape of the detection function. This is an attractive alternative to selecting among models of various shapes using, for example, AIC or other model selection criteria. Upon minimizing (8), \( \hat{\gamma} \) can then used in (7) and (6) to get full maximum likelihood estimation for all three parameters. Confidence intervals can be obtained by using profile likelihood, or approximate standard errors can be obtained using Wald-like estimates from numerical second derivatives from the log-likelihood evaluated at the maximum likelihood point estimates.

**References**

